

On Quasiminimal Excellent Classes

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Abstract

A careful exposition of Zilber’s quasiminimal excellent classes and their categoricity is given, leading to two new results: the $L_{\omega_1, \omega}(Q)$ -definability assumption may be dropped, and each class is determined by its model of dimension \aleph_0 .

Boris Zilber developed quasiminimal excellent classes in [Zil05] in order to prove that his conjectural description of complex exponentiation was uncountably categorical, that is, it has exactly one model of each uncountable cardinality. This article gives a simplified and careful exposition of quasiminimal excellent classes, and of the categoricity proof. This more careful exposition has led to two new results. We say that a quasiminimal excellent class is *degenerate* iff either it has only finite dimensional models, or it is a proper subclass of another quasiminimal excellent class. These are essentially uninteresting cases (at least in the context for which quasiminimal excellent classes were invented). In [Zil05], the proof of the existence of arbitrarily large models depended on the class being definable by an $L_{\omega_1, \omega}(Q)$ -sentence of a specific form (and on having an infinite-dimensional model). The question of whether this could be generalized to any $L_{\omega_1, \omega}(Q)$ -sentence was posed. Here we show that every nondegenerate quasiminimal excellent class is definable by an $L_{\omega_1, \omega}(Q)$ -sentence of the specific form given, hence is uncountably categorical. Furthermore, any quasiminimal excellent class with an infinite-dimensional model extends uniquely to a nondegenerate class, and hence a quasiminimal excellent class may be produced in a “bootstrap” fashion from its unique model of dimension \aleph_0 .

This article sprang from many lively and productive discussions I had with John Baldwin. The account of quasiminimal excellent classes in [Bal07] was rewritten in conjunction with this article, and incorporates many of the

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ideas given here. Some of the differences between this account of the categoricity proof and Boris Zilber's are due to John Baldwin, in particular the construction of the isomorphism in theorem 3.3 as the union of the maps f_X , and the introduction of the Shelah-style statement of excellence in lemma 3.2.

1 The Definition

Definition 1.1. A *quasiminimal excellent class* consists of the following data, satisfying axioms I, II, and III.

Data:

- For a given first-order language L , a class \mathcal{C} of L -structures.
- For each $H \in \mathcal{C}$, a function $\mathcal{P}H \xrightarrow{\text{cl}_H} \mathcal{P}H$.

Axioms:

I: Pregeometry

- I.1 For each $H \in \mathcal{C}$, cl_H is a pregeometry on H , satisfying the countable closure property (CCP): the closure of any finite set is countable.
- I.2 If $H \in \mathcal{C}$ and $X \subseteq H$, then $\text{cl}_H(X) \in \mathcal{C}$.
- I.3 If $H \in \mathcal{C}$, $X \subseteq H$, $y \in \text{cl}_H(X)$, and $f : H \rightarrow H'$ is a partial embedding with $X \cup \{y\} \subseteq \text{preim}(f)$, then $f(y) \in \text{cl}_{H'}(f(X))$.

II: \aleph_0 -homogeneity over countable models Let $H, H' \in \mathcal{C}$, let $G \subseteq H$ and $G' \subseteq H'$ be countable closed subsets or the empty set, and let $g : G \rightarrow G'$ be an isomorphism.

- II.1 If $x \in H$ and $x' \in H'$, independent from G and G' respectively, then $g \cup \{(x, x')\}$ is a partial embedding.
- II.2 If $g \cup f : H \rightarrow H'$ is a partial embedding, f has finite preimage X , and $y \in \text{cl}_H(X \cup G)$, then there is $y' \in H'$ such that $g \cup f \cup \{(y, y')\}$ is a partial embedding.

A *crown* in H is a subset $C \subseteq H$ such that there is an independent subset B of H and finitely many subsets B_1, \dots, B_n of B such that $C = \bigcup_{i=1}^n \text{cl}_H B_i$.

III: Quasiminimal excellence

Let $H, H' \in \mathcal{C}$, let C be a countable crown in H , and let $g : H \rightarrow H'$ be a closed partial embedding defined on C . For any finite subset X of $\text{cl}_H(C)$, there is a finite subset C_0 of C such that if $f : H \rightarrow H'$ has preimage X and $f \cup g|_{C_0}$ is a partial embedding then $f \cup g$ is also a partial embedding. We say that the quantifier-free type of X over C is *determined over C_0* .

Definition 1.2. We consider the class \mathcal{C} as a category by taking the *closed* embeddings, that is, those L -embeddings $H \hookrightarrow H'$ such that the image of H is closed in H' . Write $H \preceq H'$ if the inclusion $H \subseteq H'$ is a closed embedding.

A partial embedding $f : H \rightarrow H'$ is *closed* iff for every closed set X in the preimage of f , the image $f(X)$ is closed in H' .

The notion of a closed L -embedding is the right one, because it also preserves the closure operator.

Lemma 1.3. *If $H, H' \in \mathcal{C}$ with $H \subseteq H'$ and X is any subset of H then $\text{cl}_H(X) = \text{cl}_{H'}(X) \cap H$. Furthermore, if $H \preceq H'$ then $\text{cl}_H(X) = \text{cl}_{H'}(X)$. In particular, a closed embedding is a closed partial embedding (with respect to the definitions above).*

Proof. The first statement is axiom I.3 applied to the inclusion map $H \hookrightarrow H'$. For the second statement, since $X \subseteq H$ we have $\text{cl}_{H'}(X) \subseteq \text{cl}_{H'}(H) = H$, and so $\text{cl}_{H'}(X) \cap H = \text{cl}_{H'}(X)$. \square

The definition of a quasiminimal excellent class given here differs from Zilber's in the following ways. The pregeometry is included as data rather than its existence being postulated as an axiom. This avoids ambiguity where there may be more than one pregeometry to choose from. The exchange and CCP axioms are included in the definition. Early versions of Zilber's axiomatization omitted exchange, but he later realized it was necessary. In the original definition, only the models with CCP were of interest (indeed only they were quasiminimal), but for technical reasons CCP was not included as an axiom. The technical reasons are avoided by considering a quasiminimal excellent class as a category with closed embeddings (which are also introduced here). This is very natural, as it makes the class into an abstract elementary class (provided that it has unions of chains). Axiom I.3 was listed among the homogeneity axioms rather than the pregeometry axioms in [Zil05], and is here made precise. Axiom II is now \aleph_0 -homogeneity over *countable, closed* submodels, not all submodels. II.1 is weakened to consider only singletons, not independent tuples of arbitrary length. The excellence axiom III is stated only for countable models. These weakenings of

the axioms to countable models and the corresponding strengthening of the categoricity theorem are the main reason why the new results can be proved.

The terminology of a type being *determined* over a set is preferred to Zilber's original *defined* over a set, which conflicts with another, different usage. I have introduced the terminology *crown* which is shorter than *union of an independent n -system*, or *independent n -cube*, does not contain the awkward parameter n , and which I think is a better description of the concept. Zilber has also used *special subset*.

The axioms all refer to partial embeddings, and imply that the language L is rich enough to have a form of quantifier elimination (see proposition 3.5). This is a minor convenience, but is not in any way necessary. In examples, this quantifier elimination must usually be obtained by first expanding the language. It would perhaps be better to define a quasiminimal excellent class to be any class \mathcal{C}' of L' -structures for which there is an expansion by definitions to a language L , such that the resulting class \mathcal{C} satisfies the given axioms. Alternatively, one could work throughout with closed partial $L_{\omega_1, \omega}$ -maps, and adjust the axioms accordingly.

In section 4, I introduce additional axioms IV on unions of chains and the existence of an infinite-dimensional model. The subsequent analysis shows that adding these axioms to the definition of a quasiminimal excellent class rules out exactly the degenerate cases, hence it would be natural and would do no harm, but that convention is not adopted in this paper.

The reader may like to have some examples in mind. The third of the examples below is the simplest in which not all submodels are closed.

Examples 1.4. The following are quasiminimal excellent classes.

- Any strongly minimal theory in a countable language, with algebraic closure. (These are elementary classes.)
- ACF_0 , with a predicate Z and the axiom $Z(x) \longleftrightarrow \bigvee_{n \in \mathbb{Z}} x = n$, with algebraic closure (an $L_{\omega_1, \omega}$ -class).
- The theory consisting just of one equivalence relation, all of whose blocks have size \aleph_0 , the closure of a subset X being the union of the blocks meeting X (an $L_{\omega_1, \omega}(Q)$ -class).
- The motivating example: Zilber's exponential field, and the related examples of "covers of the multiplicative group" and "raising to powers".

2 Uniqueness of models up to dimension \aleph_1

Theorem 2.1. *Let \mathcal{C} be a class satisfying axioms I and II, and let $H, H' \in \mathcal{C}$ with $\dim H \leq \aleph_1$. Let $G \subseteq H$ be empty or closed and countable, and let $f_0 : G \rightarrow H'$ be a closed embedding (or the empty map if G is empty). Let B be a basis of H over G and suppose $\psi_B : H \rightarrow H'$ is an injective partial map with preimage B and image an independent set over $\text{Im } f_0$. Then $\psi := f_0 \cup \psi_B$ extends to a closed map $\hat{\psi} : H \rightarrow H'$.*

In particular, if $\text{Im } \psi$ spans H' then $\hat{\psi}$ is an isomorphism.

Proof. Well-order B as $(b_\lambda)_{\lambda < \mu}$ for some ordinal $\mu \leq \omega_1$. For each ordinal $\nu \leq \mu$, let $G_\nu = \text{cl}_H(G \cup \{b_\lambda \mid \lambda < \nu\})$. In particular, $G_0 = G$. Inductively, we construct closed maps $f_\nu : G_\nu \rightarrow H'$ such that

- $\psi|_{G_\nu} \subseteq f_\nu$, and
- If $\nu_1 \leq \nu_2$ then $f_{\nu_1} \subseteq f_{\nu_2}$.

At limit ordinals, take unions. For a successor $\nu = \lambda + 1$, we construct f_ν as $\bigcup_{n \in \mathbb{N}} h_n$ where the h_n are partial embeddings constructed inductively by the back and forth method, such that $h_0 = \{(b_\lambda, \psi(b_\lambda))\}$, and for each n ,

- $\text{preim}(h_n)$ is finite,
- $h_n \subseteq h_{n+1}$, and
- $h_n \cup f_\lambda$ is a partial embedding.

Both G_ν and $G'_\nu := \text{cl}_{H'}(f_\lambda(G_\lambda)\psi(b_\lambda))$ are countable, so list their elements in chains of length ω .

For odd n , let a be the least element of $G_\nu \setminus \text{preim } h_{n-1}$. Using the \aleph_0 -homogeneity over the countable model G_λ , there is $b \in G'_\nu$ such that, taking $h_n := h_{n-1} \cup \{(a, b)\}$, $h_n \cup f_\lambda$ is a partial embedding.

For even n , repeat the process going back rather than forth. Then f_ν is a embedding defined on all of G_ν , with image all of G'_ν , hence it is a closed embedding. It extends f_λ and $\psi|_{G_\nu}$ by construction.

Take $\hat{\psi} = \bigcup_{\lambda < \mu} f_\lambda$. This is a closed embedding as required. \square

Corollary 2.2 (Uniqueness of models up to \aleph_1). *Models of a quasiminimal excellent class of dimension less than or equal to \aleph_1 are determined up to isomorphism by their dimension. There is at most one model of cardinality \aleph_1 .* \square

A small modification of the proof explains why axiom II is called \aleph_0 -homogeneity over countable models.

Proposition 2.3. *Let \mathcal{C} be a class satisfying axioms I and II, let $H \in \mathcal{C}$, let $G \subseteq H$ be empty or countable and closed, and let \bar{x}, \bar{y} be n -tuples from H*

with the same quantifier-free type over G . Then there is an automorphism $\sigma \in \text{Aut}(H/G)$ such that $\sigma(\bar{x}) = \bar{y}$.

Proof. Write the tuple \bar{x} as $b_1, \dots, b_n, a_1, \dots, a_m$ where the b_i are independent over G and each a_i lies in $\text{cl}_H(Gb_1, \dots, b_n)$, and correspondingly write \bar{y} as $b'_1, \dots, b'_n, a'_1, \dots, a'_m$. Extend \bar{b} and \bar{b}' to bases of H over G , and let ψ_B be a bijection between the bases which sends b_i to b'_i for $i = 1, \dots, n$. Let f_0 be the identity map on G . Then follow the proof of theorem 2.1, except that in the construction of f_ν for $\nu \leq n$, start by sending a_i to a'_i for each i such that $a_i \in \text{cl}_H(Gb_1, \dots, b_\nu)$. \square

Note that the construction in the proof of theorem 2.1 cannot be carried out for H of dimension greater than \aleph_1 , because we only have \aleph_0 -homogeneity over models, not \aleph_1 -homogeneity over models. (In addition, we have only assumed homogeneity over countable models, but we show later that is not an issue, whereas most examples are actually not \aleph_1 -homogeneous over models.)

3 Uniqueness of large models

Lemma 3.1. *Let M be a set, let cl be a pregeometry on M and let B be an independent subset of M . Let $X, Y \subseteq B$. Then $\text{cl}(X) \cap \text{cl}(Y) = \text{cl}(X \cap Y)$.*

Proof. By monotonicity, $\text{cl}(X \cap Y) \subseteq \text{cl}(X)$ and $\text{cl}(X \cap Y) \subseteq \text{cl}(Y)$, so $\text{cl}(X \cap Y) \subseteq \text{cl}(X) \cap \text{cl}(Y)$.

Now suppose $z \in \text{cl}(X) \cap \text{cl}(Y) \setminus \text{cl}(X \cap Y)$. By finite character, there are $X_0 \subseteq_{\text{fin}} X$ and $Y_0 \subseteq_{\text{fin}} Y$ such that $z \in \text{cl}(X_0) \cap \text{cl}(Y_0) \setminus \text{cl}(X \cap Y)$. By exchange, there is $x \in X_0 \setminus Y$ such that $\text{cl}(X_0) = \text{cl}(X_0 z - x)$. Similarly, there is $y \in Y_0 \setminus X$ such that $\text{cl}(Y_0) = \text{cl}(Y_0 z - y)$. Hence

$$C := \text{cl}(X_0 Y_0) = \text{cl}(X_0 Y_0 z - x - y)$$

but $X_0 \cup Y_0$ is an independent set and so $\dim C = |X_0 \cup Y_0|$, and yet C is generated by a set of size $|X_0 \cup Y_0| - 1$, which is a contradiction. Hence no such z exists, and $\text{cl}(X) \cap \text{cl}(Y) = \text{cl}(X \cap Y)$ as required. \square

We translate Zilber's excellence criterion (which can be proved directly in examples) to a Shelah-style criterion.

Lemma 3.2 (Excellence – Shelah style). *Let $H, H' \in \mathcal{C}$ and let $C \subseteq H$ be a countable crown. Then every closed partial embedding $f : H \rightarrow H'$ with preimage C extends to a closed embedding $\hat{f} : \text{cl}_H(C) \hookrightarrow H'$.*

Proof. Let $\bar{C} = \text{cl}_H(C)$ and let $\bar{C}' = \text{cl}_{H'}(f(C))$. They are both countable, so choose an ordering of each of length ω .

Inductively we construction partial embeddings $f_n : H \rightarrow H'$ for $n \in \mathbb{N}$ such that for each n :

- $\text{preim}(f_n)$ is finite,
- $f_n \subseteq f_{n+1}$, and
- $f_n \cup f$ is a partial embedding.

Take $f_0 = \emptyset$. We construct the f_n for $n > 0$ via the back and forth method, going forth for odd n and back for even n .

For odd n , let a be the least element of $\bar{C} \setminus \text{preim } f_{n-1}$. Then the set $\text{preim } f_{n-1} \cup \{a\}$ is a finite subset of \bar{C} , so by axiom III (excellence) and the finite character of the pregeometry there is a finite subset C_0 of C such that the quantifier-free type of $\text{preim } f_{n-1} \cup \{a\}$ over C is determined over C_0 and $a \in \text{cl}_H(C_0)$. Let $g = f_{n-1} \cup f|_{C_0}$. By induction, $f_{n-1} \cup f$ is a partial embedding, so g is a partial embedding. By axiom II.2 (\aleph_0 -homogeneity), there is $b \in H'$ such that $f_n := g \cup \{(a, b)\}$ is a partial embedding. Since the type of $\text{preim } f_n$ over C is determined over C_0 , $f_n \cup f$ is a partial embedding, as required.

For even n , note that $f(C)$ is a crown in H' because f is a closed partial embedding. Also note that the inverse of a partial embedding is a partial embedding. Hence we can perform the same process as for odd steps, reversing the roles of H and H' , to find f_n whose image contains the least element of \bar{C}' not in the image of f_{n-1} .

Let $\hat{f} = \bigcup_{n \in \mathbb{N}} f_n$. Then \hat{f} is an embedding extending f , defined on all of \bar{C} , whose image is all of \bar{C}' . Hence \hat{f} is a closed embedding. \square

Theorem 3.3. *Let \mathcal{C} be a quasiminimal excellent class, and let $H, H' \in \mathcal{C}$. Let $G \subseteq H$ be empty or closed and let $f_0 : G \rightarrow H'$ be a closed embedding. Let B be a basis of H over G and suppose $\psi_B : H \rightarrow H'$ is an injective partial map with preimage B and image an independent set over $\text{Im } f_0$. Then $\psi := f_0 \cup \psi_B$ extends to a closed embedding $\hat{\psi} : H \rightarrow H'$.*

In particular, if $\text{Im } \psi$ spans H' then $\hat{\psi}$ is an isomorphism.

Proof. If $\dim H$ is finite then we are done by theorem 2.1, so we assume $\dim H$ is infinite. If G is empty or finite dimensional then we may extend G to a closed subset of dimension \aleph_0 , using theorem 2.1. We first prove the theorem assuming that $\dim G = \aleph_0$. The case where $\dim G > \aleph_0$ will be discussed afterwards.

For each finite subset $X \subseteq_{\text{fin}} B$, we will construct a closed embedding $f_X : \text{cl}_H(GX) \rightarrow H'$ such that

- whenever $Y \subseteq X$, we have $f_Y \subseteq f_X$, and

- $f_X \upharpoonright_X = \psi \upharpoonright_X$.

We construct the f_X by well-founded induction on the partial order of finite subsets of B . Take $f_\emptyset := f_0$.

Suppose that $X \subseteq_{\text{fin}} B$, $X \neq \emptyset$, and we already have f_Y for all proper subsets Y of X . Let $g_X = \bigcup_{Y \subsetneq X} f_Y$. Using lemma 3.1, we see that if $x \in \text{preim } f_{Y_1} \cap \text{preim } f_{Y_2}$ for $Y_1, Y_2 \subsetneq X$ then $x \in \text{preim } f_{Y_1 \cap Y_2}$, and so by hypothesis $f_{Y_1}(x) = f_{Y_2}(x)$. Hence g_X is a well-defined partial function. We must show that g_X is a partial embedding.

Say $X = \{x_1, \dots, x_n\}$, and for $i = 1, \dots, n$ let $Y_i = X \setminus \{x_i\}$. Let $C_k = \bigcup_{i=1}^k \text{cl}_H(GY_i)$ and let $h_k = \bigcup_{i=1}^k f_{Y_i}$. So $g_X = h_n$.

We prove by induction on k that h_k is a partial embedding. The $k = 1$ case is immediate. For the induction step, take tuples $a \in C_{k-1}$ and $b \in \text{cl}_H(GY_k)$. We construct an automorphism of a submodel of H' which has the effect of moving the parameters a inside $\text{cl}_H(GY_k)$. Let $A \subseteq_{\text{fin}} G$ such that $a, b \in \text{cl}_H(AX)$, and let $G_0 = \text{cl}_H(A)$. Let $z \in G \setminus G_0$, let $H_0 = \text{cl}_H(G_0Xz)$, and let $H'_0 = \text{cl}_{H'}(\psi(G_0Xz))$. By theorem 2.1, there is an automorphism σ of H_0 , fixing $\text{cl}_H(G_0Y_k)$ and swapping x_k and z .

The idea is to compare h_k on H_0 with the composite embedding $\tau = \sigma'^{-1}f_{Y_k}\sigma$, where σ' is the automorphism of H'_0 which “corresponds” to σ . To have a notion of what “corresponds” means, we must choose a suitable isomorphism between H_0 and H'_0 . Fortunately, the condition of being “suitable” is weaker than the compatibility condition we are trying to prove!

Let e be a closed embedding $e : \text{cl}_H(GX) \rightarrow H'$ extending $h_{k-1} \cup \psi \upharpoonright_X$. For $k = 2$, such an e exists because $X = Y_1 \cup \{x_1\}$, so $h_1 \cup \{(x_1, \psi(x_1))\}$ extends to some e by theorem 2.1. For $k \geq 3$, C_{k-1} is a crown whose closure is $\text{cl}_H(GX)$, so h_k extends to some e by lemma 3.2. Let $\sigma' = e\sigma e^{-1}$, and let $\tau = \sigma'^{-1}f_{Y_k}\sigma = e\sigma^{-1}e^{-1}f_{Y_k}\sigma$.

Write Y_{ik} for $Y_i \cap Y_k$. The tuple $a \in \bigcup_{i=1}^{k-1} \text{cl}_H(G_0Y_i)$, so

$$\sigma(a) \in \bigcup_{i=1}^{k-1} \text{cl}_H(G_0Y_{ik}z) \subseteq \bigcup_{i=1}^{k-1} \text{cl}_H(GY_{ik}) \subseteq \text{preim } h_{k-1}.$$

By hypothesis, f_{Y_k} agrees with f_{Y_i} on $\text{cl}_H(GY_{ik})$, hence f_{Y_k} agrees with h_{k-1} on $\bigcup_{i=1}^{k-1} \text{cl}_H(GY_{ik})$. Also e and h_k both extend h_{k-1} , so

$$\tau(a) = e\sigma^{-1}e^{-1}f_{Y_k}\sigma(a) = h_{k-1}\sigma^{-1}h_{k-1}^{-1}h_{k-1}\sigma(a) = h_{k-1}(a) = h_k(a).$$

The tuple $b \in \text{cl}_H(G_0Y_k)$, so it is fixed by σ . The embeddings f_{Y_k} and e preserve the closure, so $e^{-1}f_{Y_k}(b)$ is fixed by σ^{-1} . So

$$\tau(b) = e\sigma^{-1}e^{-1}f_{Y_k}\sigma(b) = ee^{-1}f_{Y_k}(b) = f_{Y_k}(b) = h_k(b).$$

Thus for any quantifier-free formula R ,

$$H \models R(a, b) \iff H' \models R(\tau(a), \tau(b)) \iff H' \models R(h_k(a), h_k(b)).$$

This holds for any tuples a, b (for a suitable choice of τ) and so h_k is a partial embedding. In particular, $g_X = h_n$ is a partial embedding. It is a union of finitely many closed partial embeddings, hence is a closed partial embedding.

By lemma 3.2, g_X extends to a closed embedding $f_X : \text{cl}_H(GX) \rightarrow H'$. Thus, by induction, we have compatible embeddings f_X for every $X \subseteq_{\text{fin}} B$. Let $\hat{\psi} = \bigcup_{X \subseteq_{\text{fin}} B} f_X$, a closed embedding. By the finite character of the closure, $H = \bigcup_{X \subseteq_{\text{fin}} B} \text{cl}_H(GX)$. Hence $\hat{\psi}$ is a total map on H . That completes the proof in the case where $\dim G \leq \aleph_0$.

If $\dim G > \aleph_0$, let G' be a closed subset of G of dimension \aleph_0 , and let B_1 be a basis of G over G' . Inductively construct embeddings f_X as before, with G' in place of G and $B \cup B_1$ in place of B , except that for $X \subseteq_{\text{fin}} B_1$ take $f_X = f_0|_{\text{cl}_H(G'X)}$. Then the map $\hat{\psi}$ obtained will extend ψ as required. \square

Corollary 3.4 (Uniqueness of models). *Models of a quasiminimal excellent class are determined up to isomorphism by their dimension. There is at most one model of any uncountable cardinality.* \square

A small modification of the proof of theorem 3.3, similar to the case of proposition 2.3, gives the following full homogeneity and quantifier elimination statements.

Proposition 3.5. *Let \mathcal{C} be a quasiminimal excellent class.*

- *Any model in \mathcal{C} is \aleph_0 -homogeneous over the empty set and over every closed submodel.*
- *The Galois types of finite tuples over the empty set and over closed submodels are equal to the quantifier-free L -types.* \square

4 Existence of models

The axioms for a quasiminimal excellent class do not imply that any models exist at all. If there is a model of dimension κ , then for any cardinal $\lambda < \kappa$ there is a model of dimension λ , by axiom I.2. Hence the models of a quasiminimal excellent class are indexed by some initial segment of the class of cardinals. There is nothing in the axioms to say that we cannot have a proper initial segment – indeed any initial segment of any quasiminimal excellent class is also a quasiminimal excellent class. There are also quasiminimal excellent classes with only models of finite dimension. To exclude these degenerate cases, we consider additional axioms. We will see that they exclude only these degenerate cases.

IV: Chains

IV.1 The category \mathcal{C} has unions of chains of all ordinal lengths. That is, suppose $(H_\mu)_{\mu < \lambda}$ is an ordinal-indexed chain of models of \mathcal{C} with closed embeddings. Let H be the union of the chain (as an L -structure), and for $X \subseteq H$, define $cl_H(X) = \bigcup_{\mu < \lambda} cl_{H_\mu}(X \cap H_\mu)$. Then $\langle H, cl_H \rangle \in \mathcal{C}$.

IV.2 \mathcal{C} contains an infinite dimensional model.

It is easy to show:

Proposition 4.1. *A quasiminimal excellent class which satisfies axiom IV.1 (unions of chains) is an abstract elementary class with Löwenheim cardinal at most \aleph_0 .* \square

Theorem 4.2 (Existence of models). *Let \mathcal{C} be a quasiminimal excellent class satisfying axiom IV. Then for every cardinal κ there is a unique $H \in \mathcal{C}$ of dimension κ . In particular, \mathcal{C} is uncountably categorical. Conversely, any uncountably categorical quasiminimal excellent class satisfies axiom IV.*

Proof. We have already proved uniqueness, and it remains to prove existence. By induction on ordinals λ we construct the initial segment of length λ of a chain $(M_\mu)_{\mu \in \text{ON}}$ in \mathcal{C} where each M_μ has a chosen basis indexed by μ , and the inclusion maps $M_\mu \hookrightarrow M_\nu$ of the chain extend the inclusion maps $\mu \hookrightarrow \nu$ of the ordinals (identified with the chosen bases). Here ON is the ordered class of ordinals.

By axiom IV.2, there is $M \in \mathcal{C}$ with $\dim M = \kappa$, for some infinite cardinal κ . Choose an ordering of a basis of M of length the initial ordinal α of size κ . Taking the closures of the initial segments of this basis gives the chain $(M_\mu)_{\mu < \alpha}$ in \mathcal{C} .

There are two cases for the inductive step. If λ is a successor $\lambda = \nu^+$ then we already have a model (for example M_ν) of dimension $|\lambda|$. We choose a new ordering of length λ for a basis, and, using theorem 3.3, we choose a closed embedding $M_\nu \hookrightarrow M_\lambda$ extending the inclusion of bases.

If λ is a limit ordinal then by induction we have a chain $(M_\mu)_{\mu < \lambda}$ in \mathcal{C} . By axiom IV.1, the union of this chain lies in \mathcal{C} , and the inclusion maps of the M_μ into the limit are closed. It has a basis indexed by λ , so we take it as M_λ . That gives one direction.

For the converse, suppose \mathcal{C} is uncountably categorical, and $(H_\nu)_{\nu < \lambda}$ is a chain in \mathcal{C} . Let κ be a cardinal which is an upper bound for the dimensions of the H_ν , and for $|\lambda|$, and let α be the initial ordinal of κ . Then the model M_κ in \mathcal{C} of dimension κ can be written as the union of a chain $(M_\mu)_{\mu < \alpha}$, as above. The chain $(H_\nu)_{\nu < \lambda}$ is isomorphic to a subchain of $(M_\mu)_{\mu < \alpha}$ (possibly

with repeats), so its union is naturally a closed subset of M_κ . Hence the union lies in \mathcal{C} . Thus \mathcal{C} satisfies IV.1. Axiom IV.2 is immediate. \square

5 Definability

This section is mainly about the definability of a quasiminimal excellent class, but we first give a result about definable sets within a class. The motivation for the word *quasiminimal* is that every definable set is countable or cocountable.

Lemma 5.1 (Quasiminimality). *Let \mathcal{C} be a quasiminimal excellent class, and let $H \in \mathcal{C}$.*

- *If $X \subseteq H$ and $a, b \in H \setminus \text{cl}_H(X)$ then there is $\sigma \in \text{Aut}(H/X)$ such that $\sigma(a) = b$.*
- *Every subset of H which is definable with countably many parameters in any logic respecting L -automorphisms (for example $L_{\infty, \omega}$ or $L_{\omega_1, \omega}(Q)$) is either countable or cocountable.*

Proof. The first part is immediate from theorem 3.3. For the second part, let $S \subseteq H$ be definable from a countable parameter set A . Then $H \setminus \text{cl}_H(A)$ is a single orbit of $\text{Aut}(H/A)$, and is cocountable, and either S does not meet $H \setminus \text{cl}_H(A)$ or S contains $H \setminus \text{cl}_H(A)$. \square

There is no assumption that the language L should be countable, but in fact we may assume that it is.

Proposition 5.2. *Let \mathcal{C} be a quasiminimal excellent class, in a language L . Then there is a countable sublanguage L^0 of L such that the class \mathcal{C}^0 of reducts to L^0 of models in \mathcal{C} is also a quasiminimal excellent class, and the reduct map $\mathcal{C} \rightarrow \mathcal{C}^0$ is an isomorphism of categories.*

Proof. Let M be a countable model, G a closed submodel of M or \emptyset , and a, b be n -tuples from M , and suppose the quantifier-free types $\text{qftp}(a/G)$ and $\text{qftp}(b/G)$ are different. Then there is a symbol from the signature of L which witnesses this. Up to isomorphism, there are only countably many such tuples $\langle M, G, a, b \rangle$. Hence there is a countable sublanguage L^0 of L which witnesses all such differences of quantifier-free types. By induction using \aleph_0 -homogeneity over models, partial L^0 -embeddings and partial L -embeddings of countable models coincide.

The language L is finitary (that is, every symbol has finite arity), so a map between two models is an L -embedding precisely when its restriction to every finitely generated substructure is an L -embedding. By the countable

closure property, an embedding is closed precisely when its restriction to every countable submodel is closed. Hence partial (and total) closed L^0 -embeddings and closed L -embeddings of any models coincide. In particular, quantifier-free L^0 -types coincide with quantifier-free L -types. Hence for each relation symbol R of L , there is an quantifier-free L^0 -formula $\theta_R(\bar{x})$ such that for any $M \in \mathcal{C}$, $M \models (\forall \bar{x})[R(\bar{x}) \leftrightarrow \theta_R(\bar{x})]$. This gives a unique way of expanding a model in \mathcal{C}^0 to a model in \mathcal{C} , which in turn gives an inverse to the reduct map. \square

Lemma 5.3 ($L_{\omega_1, \omega}$ -definability of the pregeometry). *Let \mathcal{C} be a quasiminimal excellent class. For each $n \in \mathbb{N}$ there is a quantifier-free $L_{\omega_1, \omega}$ -formula $\pi_n(x, y_1, \dots, y_n)$ such that for each $H \in \mathcal{C}$ and each $a, b_1, \dots, b_n \in H$, we have*

$$a \in \text{cl}_H(\bar{b}) \text{ iff } H \models \pi_n(a, \bar{b}).$$

Proof. By proposition 5.2 we may assume the language is countable. For each $n \in \mathbb{N}$, let M_n be the model of dimension n (if it exists, and the model of maximum dimension if it does not). For each n , every quantifier-free n -type is realised in the closure of its realization, hence in M_{n+1} , hence there are only countably many quantifier-free n -types over \emptyset . The types are given by quantifier-free $L_{\omega_1, \omega}$ -formulas, say $(\beta_j(\bar{y}))_{j \in \mathbb{N}}$. By the same argument, for each j and \bar{b} of type β_j , there are only countably many 1-types of elements in the closure of \bar{b} , and their types are also given by $L_{\omega_1, \omega}$ -formulas, say $(\gamma_{ij}(x, \bar{b}))_{i \in \mathbb{N}}$. So the formula

$$\pi_n(x, \bar{y}) \equiv \bigvee_{j \in \mathbb{N}} \left(\beta_j(\bar{y}) \wedge \bigvee_{i \in \mathbb{N}} \gamma_{ij}(x, \bar{y}) \right)$$

works for M_{n+1} . If $H \in \mathcal{C}$ and $a, b_1, \dots, b_n \in H$, then either there is a closed embedding $H \hookrightarrow M_{n+1}$, or there is a closed embedding $M_{n+1} \hookrightarrow H$ whose image contains a, b_1, \dots, b_n . In each case, $H \models \pi_n(a, \bar{b})$ iff $M_{n+1} \models \pi_n(a, \bar{b})$ since L -embeddings preserve quantifier-free $L_{\omega_1, \omega}$ -formulas, and $a \in \text{cl}_H(\bar{b})$ iff $a \in \text{cl}_{M_{n+1}}(\bar{b})$ by lemma 1.3. Hence the same formulas π_n work for every $H \in \mathcal{C}$. \square

Lemma 5.4. *Let \mathcal{C} be a quasiminimal excellent class, let $f : M \rightarrow N$ be a closed embedding in \mathcal{C} and suppose that $\dim M \geq \aleph_0$. Then f is an $L_{\omega_1, \omega}$ -embedding.*

Proof. We prove by induction on formulas that for any $L_{\omega_1, \omega}$ -formula $\theta(\bar{x})$ and any $\bar{a} \in M$, we have $M \models \theta(\bar{a}) \iff N \models \theta(\bar{a})$.

f is an L -embedding, so atomic formulas are preserved.

The cases $\theta(\bar{x}) \equiv \bigwedge_{i \in I} \theta_i(\bar{x})$ and $\theta(\bar{x}) \equiv \neg \theta_0(\bar{x})$ are immediate.

If $\theta(\bar{x}) \equiv \exists y \varphi(y, \bar{x})$ then the left to right case is immediate. Suppose $N \models \exists y \varphi(y, \bar{a})$. Then for some $b \in N$, $N \models \varphi(b, \bar{a})$. If $b \in M$ then we are done. Suppose $b \notin M$. Since $\text{cl}(\bar{a})$ is finite dimensional, we can choose $c \in M \setminus \text{cl}(\bar{a})$. The point b is independent from M , so by lemma 5.1 there is an automorphism of N fixing $\text{cl}(\bar{a})$ and swapping b and c . So $N \models \varphi(c, \bar{a})$ and, by the inductive hypothesis, $M \models \varphi(c, \bar{a})$. Hence $M \models \exists y \varphi(y, \bar{a})$.

Hence f is an $L_{\omega_1, \omega}$ -embedding as required. \square

It is easy to extend the proof to show that if $\dim H \geq \aleph_1$ then f is an $L_{\omega_1, \omega}(Q)$ -embedding.

Theorem 5.5. *Let \mathcal{C} be a quasiminimal excellent class in a countable language, with an infinite dimensional model. For each $n \leq \omega$, let M_n be the model of dimension n , and let σ_n be its Scott sentence. Let*

$$\Sigma = \left[\bigvee_{n \leq \omega} \sigma_n \right] \wedge \left[\bigwedge_{n \in \mathbb{N}} (\forall y_1 \dots, y_n) \neg (Qx) \pi_n(x, y_1, \dots, y_n) \right]$$

where Q is the quantifier “there exist uncountably many”. Then $\text{Mod}(\Sigma)$ is an uncountably categorical quasiminimal excellent class containing \mathcal{C} . Furthermore, if \mathcal{C} satisfies axiom IV then $\mathcal{C} = \text{Mod}(\Sigma)$.

Proof. We check axioms I—IV for $\text{Mod}(\Sigma)$. The statement that the π_n define a pregeometry can be axiomatized as an $L_{\omega_1, \omega}$ -sentence, and it is true in each M_n , hence it follows from each σ_n . The countable closure property is explicit in Σ , hence axiom I.1 holds. Axiom I.3 holds because the pregeometry is defined by the quantifier-free $L_{\omega_1, \omega}$ -formulas π_n . From these axioms we get the notion of dimension for each model of Σ , and we also get the notion of a closed embedding.

For each uncountable cardinal κ , let $\mathcal{D}_\kappa = \{N \models \Sigma \mid \dim N < \kappa\}$. Axioms I.1 and I.3 hold for each \mathcal{D}_κ as well.

Axioms II and III are statements about the countable models. Any countable model of Σ must be one of the M_n for $n \leq \omega$, since Scott sentences are \aleph_0 -categorical. Thus the countable models of Σ are just the countable models of \mathcal{C} . Hence $\text{Mod}(\Sigma)$ and each \mathcal{D}_κ satisfy II and III. The infinite-dimensional model M_ω lies in $\text{Mod}(\Sigma)$ and each \mathcal{D}_κ , so IV.2 holds. It remains to prove axioms I.2 and IV.1. We prove two related families of statements:

- 1 $_\kappa$) \mathcal{D}_κ satisfies I.2 and hence is a quasiminimal excellent class.
- 2 $_\kappa$) If $H \models \Sigma$ and N is a closed subset of H with $\dim N = \kappa$ then $N \models \Sigma$.

We must prove statement 2 for all κ , and statement 1 for uncountable κ . First we prove statement 2 for countable κ . For $n \in \mathbb{N}$, let x_1, \dots, x_n be variables not occurring in σ_n , and let $\sigma'_n(x_1, \dots, x_n)$ be the $L_{\omega_1, \omega}$ -formula obtained from σ_n by recursively replacing all quantified subformulas of the form $\exists y[\varphi(y)]$ by $\exists y[\pi_n(y, x_1, \dots, x_n) \wedge \varphi(y)]$, for any subformula $\varphi(y)$ (and similarly for universal quantifiers). Let $\text{Indep}_n(x_1, \dots, x_n)$ be the formula $\bigwedge_{i=1}^n \neg \pi_{i-1}(x_i, x_1, \dots, x_{i-1})$. Then $\text{Indep}_n(x_1, \dots, x_n)$ says that the x_i are cl-independent, and $\sigma'_n(x_1, \dots, x_n)$ says that the closure of the x_i is a model of σ_n . Thus for each $m \leq \omega$, and each $n \in \mathbb{N}$,

$$\sigma_m \vdash (\forall x_1, \dots, x_n)[\text{Indep}_n(x_1, \dots, x_n) \rightarrow \sigma'_n(x_1, \dots, x_n)].$$

Hence if N is finite dimensional, it is a model of Σ .

Now suppose $\dim N = \aleph_0$. Let $(b_n)_{n < \omega}$ be a basis for N , and let $N_m = \text{cl}_H(\{b_n \mid n < m\})$ for each $m < \omega$. Then N is the union of the chain $(N_m)_{m < \omega}$ and, by the above, $N_m \cong M_m$. The union of the chain $(M_m)_{m \leq \omega}$ is M_ω , hence $N \cong M_\omega$. In particular, $N \models \Sigma$. Thus 2_κ holds for all countable κ .

Now we prove statements 1_κ and 2_κ together for uncountable κ , by induction. Suppose inductively that 2_λ holds for every $\lambda < \kappa$. Then if $H \in \mathcal{D}_\kappa$, every closed subset of H has dimension less than κ , so is a model of Σ . Hence 1_κ holds.

Now suppose $H \models \Sigma$ and N is a closed subset of H with $\dim N = \kappa$. Identifying κ with its initial ordinal, let $(b_\lambda)_{\lambda < \kappa}$ be a basis of N and let $N_\mu = \text{cl}_H(\{b_\lambda \mid \lambda < \mu\})$. Then N is the union of the chain $(N_\mu)_{\omega \leq \mu < \kappa}$, and by induction each N_μ models Σ . The chain lies in \mathcal{D}_κ , which by the same induction hypothesis satisfies I.2 and hence is a quasiminimal excellent class. Thus by lemma 5.4, the chain is an $L_{\omega_1, \omega}$ -chain. Each N_μ in the chain is infinite dimensional, thus models σ_ω , and hence $N \models \sigma_\omega$. Also $N \subseteq H$, and H has the CCP, so N also has the CCP. Thus $N \models \Sigma$, that is, 2_κ holds. Thus, by induction, 1_κ and 2_κ hold for all uncountable κ .

Thus $\text{Mod}(\Sigma)$ satisfies axiom I.2, and so is a quasiminimal excellent class. If $(H_\lambda)_{\lambda < \kappa}$ is any chain in $\text{Mod}(\Sigma)$ then either its union is finite dimensional (and lies in $\text{Mod}(\Sigma)$) or the chain is eventually infinite dimensional and by lemma 5.4 is eventually an $L_{\omega_1, \omega}$ -chain. In the latter case, as above, the union H of the chain is a model of σ_ω . If X is a finite subset of H then $X \subseteq H_\lambda$ for some $\lambda < \kappa$, and $\text{cl}_H(X) = \text{cl}_{H_\lambda}(X)$. Since H_λ has the CCP, this closure is countable. Hence H has the CCP, and so $H \models \Sigma$. Thus $\text{Mod}(\Sigma)$ satisfies axiom IV.1. By theorem 4.2, $\text{Mod}(\Sigma)$ is an uncountably categorical quasiminimal excellent class.

If $H \in \mathcal{C}$ then either H is finite dimensional in which case H is M_n for some $n \in \mathbb{N}$, or there is a closed embedding $M_\omega \hookrightarrow H$. By lemma 5.4, this

embedding is an $L_{\omega_1, \omega}$ -embedding, so $H \models \sigma_\omega$. In either case, $H \models \Sigma$. So $\mathcal{C} \subseteq \text{Mod}(\Sigma)$. Since \mathcal{C} satisfies I.2, it is an initial segment of $\text{Mod}(\Sigma)$. That is, either $\mathcal{C} = \mathcal{D}_\kappa$ for some κ or \mathcal{C} satisfies IV.1 and $\mathcal{C} = \text{Mod}(\Sigma)$. \square

Corollary 5.6. *Let \mathcal{C} be a quasiminimal excellent class with a model of dimension \aleph_0 . Then there is a unique quasiminimal excellent class \mathcal{C}' , containing \mathcal{C} , which is uncountably categorical.*

Proof. By proposition 5.2 we may assume the language is countable. Then theorem 5.5 gives $\text{Mod}(\Sigma)$ as one such \mathcal{C}' . If \mathcal{C}'' is any uncountably categorical quasiminimal excellent class containing \mathcal{C} then by theorem 4.2 it satisfies axiom IV, so by theorem 5.5 again it is equal to $\text{Mod}(\Sigma)$. Hence \mathcal{C}' is unique. \square

From this result we see that nothing would be lost by adding axiom IV to the definition of a quasiminimal excellent class, unless perhaps there can be interesting behaviour of finite dimensional models. See the questions at the end of this paper.

In theorem 5.5, the sentence Σ depends only on the model M_ω , because the M_n are substructures of M_ω . We can extract the properties of M_ω which are needed to produce a quasiminimal excellent class.

Corollary 5.7. *Let M be a countable L -structure, equipped with a pregeometry cl , satisfying the following axioms.*

I' (Pregeometry)

The pregeometry is quantifier-free $L_{\omega_1, \omega}$ -definable, and $\dim M = \aleph_0$.

II' (\aleph_0 -homogeneity over closed sets)

Let $G \subseteq M$ be closed or empty.

II.1' If $x, x' \in M$ are each independent from G , then $\text{qftp}(x/G) = \text{qftp}(x'/G)$.

II.2' Let \bar{x}, \bar{x}' be finite tuples from M such that $\text{qftp}(\bar{x}/G) = \text{qftp}(\bar{x}'/G)$, and let $y \in M$. Then there is $y' \in M$ such that $\text{qftp}(\bar{x}y/G) = \text{qftp}(\bar{x}'y'/G)$.

III' (Excellence)

If C is a crown in M and \bar{x} is a finite tuple from $\text{cl}(C)$, then there is a finite subset C_0 of C such that for any tuple \bar{x}' ,

$$\text{qftp}(\bar{x}/C_0) = \text{qftp}(\bar{x}'/C_0) \implies \text{qftp}(\bar{x}/C) = \text{qftp}(\bar{x}'/C).$$

Then there is a unique quasiminimal excellent class \mathcal{C} which satisfies axiom IV such that $\langle M, \text{cl} \rangle \in \mathcal{C}$.

Proof. Let \mathcal{C}_0 be the class of L -structures consisting of M and all its cl -closed substructures, equipped with the pregometry cl and its restrictions. The axioms I'—III' ensure that \mathcal{C}_0 is a quasiminimal excellent class with a model of dimension \aleph_0 , and then corollary 5.6 gives the unique class \mathcal{C} . \square

6 Questions

We conclude with some further questions.

1. Is there a class which satisfies axioms I and II, but not III? It seems likely that there is, but I do not believe that any examples are currently known.
2. Does axiom III follow from lemma 3.2? The former is Zilber's definition of excellence and the latter should be Shelah's with respect to this abstract elementary class, albeit with prime model in place of primary model.
3. If \mathcal{C} is a quasiminimal excellent class with models of arbitrarily large finite dimension (and hence of all finite dimensions), there is a well-defined L -structure M constructed as the union of all of the finite dimensional models. Is $\mathcal{C} \cup \{M\}$ necessarily a quasiminimal excellent class?
4. What quasiminimal excellent classes are there with models only of dimension up to some finite n , with $n \geq 1$?

References

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